

SCALING LIMITS OF RANDOM NORMAL MATRIX PROCESSES AT SINGULAR BOUNDARY POINTS

YACIN AMEUR, NAM-GYU KANG, NIKOLAI MAKAROV, AND ARON WENNMANN

ABSTRACT. We study scaling limits for random normal eigenvalue ensembles near certain types of singular points of the droplet. In particular, we prove existence of new types of determinantal point fields, which differ from those which can appear at a regular boundary point.

The method of rescaled Ward identities was introduced in the paper [3], where the main focus was on the case of scaling limits at a regular boundary point of the droplet. In this note, we will apply the same method to obtain results about scaling limits near certain types of *singular* boundary points, i.e. cusps and double points.

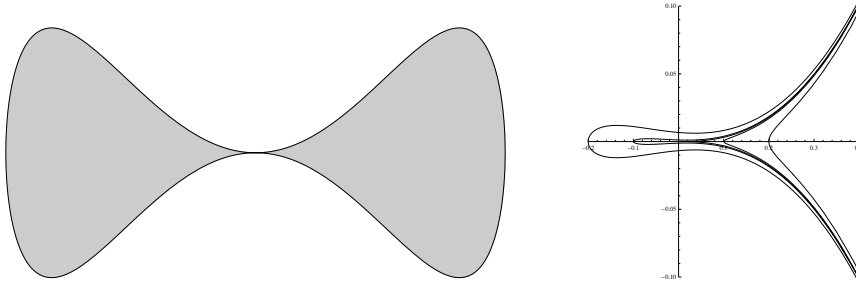


FIGURE 1. A droplet with a double point. The picture on the right shows a (5,2)-cusp under Laplacian growth/Hele-Shaw evolution.

Recall that, in the random normal matrix model, we are given a suitable real-valued function Q , called the "potential". We consider configurations (or "systems") $\{\zeta_j\}_1^n$ of points in \mathbb{C} , having the interpretation of identical point charges subject to the external field Q . The energy of the system is defined to be

$$H_n(\zeta_1, \dots, \zeta_n) = \sum_{j \neq k} \log \frac{1}{|\zeta_j - \zeta_k|} + n \sum_{j=1}^n Q(\zeta_j),$$

and we consider the ensemble of systems picked randomly with respect to the Boltzmann-Gibbs law,

$$(0.1) \quad d\mathbb{P}_n(\zeta) = \frac{1}{Z_n} e^{-H_n(\zeta)} dV_n(\zeta), \quad \zeta = (\zeta_1, \dots, \zeta_n) \in \mathbb{C}^n.$$

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Here dV_n is Lebesgue measure in \mathbb{C}^n divided by π^n , and $Z_n = \int e^{-H_n} dV_n$.

As $n \rightarrow \infty$, the random samples $\{\zeta_j\}_1^n$ tend to condensate on a compact set S known as the droplet, the boundary of which is a finite union of real-analytic arcs, and possibly containing finitely many singular points.

The main problem considered in this note is the microscopic properties of the system near certain types of singular boundary point $p \in \partial S$, as $n \rightarrow \infty$.

We prove that nothing of interest is to be found in the vicinity of the singular point itself, but instead we shall find new point fields located somewhat inside the droplet, close to the singular point. This is accomplished by zooming about a "moving point", which approaches the singular point at a proper rate. Cf. Figure 1.

1. INTRODUCTION AND MAIN RESULTS

Notational conventions. We write $D(p; r)$ for the disk centered at p of radius \mathbb{R} . \mathbb{C}_+ denotes the upper half-plane. The characteristic function of a set E will be denoted χ_E . We use the notation $\Delta = \partial\bar{\partial}$, so Δ is 1/4 of the usual Laplacian. We write $dA(z) = d^2z/\pi$ for two-dimensional Lebesgue measure in \mathbb{C} divided by π . A continuous function $f : \mathbb{C}^2 \rightarrow \mathbb{C}$ is termed *Hermitian* if $f(z, w) = \overline{f(w, z)}$. We say that f is *Hermitian-entire* if f is Hermitian and entire as a function of z and \bar{w} . A Hermitian-entire function is uniquely determined by its diagonal values $f(z, z)$. A Hermitian function c is called a *cocycle* if $c(z, w) = g(z)\overline{g(w)}$ for a continuous unimodular function g .

1.1. External potential and droplet. Our basic setup is like in [3]. In short, let $Q : \mathbb{C} \rightarrow \mathbb{R} \cup \{\infty\}$ is a suitable "external potential" of sufficient growth; requiring $\liminf_{\zeta \rightarrow \infty} Q(\zeta)/\log|\zeta|^2 > 1$ will do. If μ is a positive, compactly supported Borel measure we define its weighted logarithmic Q -energy by

$$(1.1) \quad I_Q[\mu] = \int_{\mathbb{C}} Q d\mu + \int_{\mathbb{C}^2} \log \frac{1}{|\zeta - \eta|} d\mu(\zeta) d\mu(\eta).$$

A theorem of Frostman asserts that there is a unique *equilibrium measure* σ of total mass 1, which minimizes $I_Q[\mu]$ over all compactly supported Borel probability measures μ on \mathbb{C} . The support of the measure σ is called the *droplet* in external field Q , and is denoted

$$S = S[Q] := \text{supp } \sigma.$$

It is well-known (see e.g. [15]) that S is a compact set and σ is absolutely continuous and given by the formula

$$(1.2) \quad d\sigma(z) = \Delta Q(z) \chi_S(z) dA(z).$$

Here and throughout we make the convention that $\Delta := \partial\bar{\partial}$ is 1/4 of the standard Laplacian while $dA(z) := d^2z/\pi$ is two-dimensional Lebesgue measure divided by π .

We make the standing assumptions that there is a neighbourhood Ω of S such that

- (1) Q is real-analytic in Ω ,
- (2) $\Delta Q > 0$ in Ω .

Under these conditions, the complement S^c has a local Schwarz function, and one can apply Sakai's regularity theorem [16] to conclude that the boundary points $p \in \partial S$ can be characterized as follows.

The most common type of boundary point is a *regular* boundary point. This is a point p such that there exists a neighbourhood $D = D(p; \epsilon)$ such that $D \setminus S$ is a Jordan domain and $D \cap (\partial S)$ is a simple real-analytic arc. By Sakai's theorem, all but finitely many boundary points of S are regular. The finitely many exceptional points are called *singular*.

When analyzing a singular point, we can *w.l.o.g.* assume that it is located on the *outer* boundary of S , i.e. on the boundary of the unbounded component U of $\hat{\mathbb{C}} \setminus S$. If there are other boundary components, they can be treated in the same way.

There are two kinds of singular boundary points. A point at the outer boundary $p \in \partial U$ is a (conformal) *cusp* if there is $D = D(p; \epsilon)$ such that $D \setminus S$ is a Jordan domain and every conformal map $\Phi : \mathbb{C}_+ \rightarrow D \setminus U$ with $\Phi(0) = p$ extends analytically to a neighbourhood of 0 and satisfies $\Phi'(0) = 0$; p is a *double point* if there is a disk D about p such that $D \setminus S$ is a union of two Jordan domains, and p is a regular boundary point of each of them. Note that the cusps which appear on the boundary of a droplet S always point out from S .

One can further classify singular points according to degrees of tangency; we briefly recall how this works for cusps.

Thus assume that ∂S has a cusp at the outer boundary ∂U at $p = 0$. We can assume that a conformal map $\Phi : \mathbb{C}_+ \rightarrow U$ satisfies

$$\Phi'(z) = z + a_2 z^2 + \cdots + (a_{\nu-1} + ib)z^{\nu-1} + \cdots$$

where a_j and b are real and $b \neq 0$. This means that

$$\Phi(z) = \frac{1}{2}z^2 + \frac{a_2}{3}z^3 + \cdots + \frac{a_{\nu-1} + ib}{\nu}z^{\nu} + \cdots.$$

If we write $\Phi = u + iv$, this gives

$$u(x) = \frac{1}{2}x^2 + \cdots, \quad v(x) = \frac{b}{\nu}x^{\nu} + \cdots, \quad (x \in \mathbb{R}).$$

By definition, this means that the cusp at 0 is of type $(\nu, 2)$.

Some cusps, in particular $(3, 2)$ -cusps, which are generic in Sakai's theory can not appear on a free boundary. By contrast, the methods in this paper do not apply to $(3, 2)$ -cusps, but we are instead able to treat $(\nu, 2)$ cusps where ν is odd and $\nu \geq 5$, provided that our standing assumptions (1) and (2) hold.¹ We will also obtain results for double points.

Droplets with singular boundary points have been studied e.g. in the papers [7, 13, 20], the book [16], and the thesis [8]. We refer to those sources and the references there for more detailed information on various types of singular boundary points.

1.2. Rescaled ensembles. Let $\{\zeta_j\}_1^n$ be a random sample from the Boltzmann-Gibbs distribution (0.1). As in [3], we will denote by boldface characters the objects pertaining to this ensemble. We shall for example use the k -point function

$$\mathbf{R}_{n,k}(\eta_1, \dots, \eta_k) = \lim_{\varepsilon \rightarrow 0} [\varepsilon^{-2k} \cdot \mathbb{P}_n(\cap_{j=1}^k \{N_{D(\eta_j; \varepsilon)} \geq 1\})],$$

where N_B is the number of points ζ_j which fall in the set B . It is well-known that we can write

$$\mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k) = \det(\mathbf{K}_n(\zeta_i, \zeta_j))_{i,j=1}^k,$$

where \mathbf{K}_n is a Hermitian function, which we call a correlation kernel of the process.

¹In fact it is enough to require that they hold locally, near the singular point in question.

Now consider a moving point $p_n \in S$ and a sequence of angles $\theta_n \in \mathbb{R}$. We shall consider rescaled point processes $\Theta_n = \{z_j\}_1^n$ where

$$(1.3) \quad z_j = e^{-i\theta_n} \sqrt{n\Delta Q(p_n)} (\zeta_j - p_n).$$

The law of the process Θ_n is defined as the image of the Boltzmann-Gibbs law under the map $\{\zeta_j\}_1^n \mapsto \{z_j\}_1^n$.

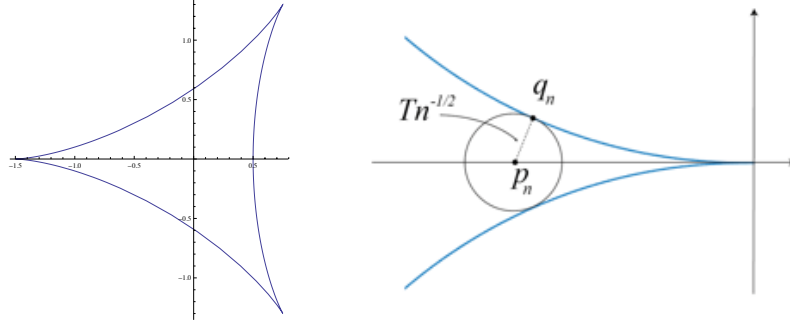


FIGURE 2. The deltoid has three maximal $(3, 2)$ -cusps. The picture on the right shows a moving point p_n near a cusp.

Objects pertaining to the rescaled system $\{z_j\}_1^n$ are denoted by plain symbols. For example, the k -point function of the rescaled system will be written

$$R_{n,k}(z_1, \dots, z_k) := \frac{1}{(n\Delta Q(p_n))^k} \mathbf{R}_{n,k}(\zeta_1, \dots, \zeta_k).$$

The rescaled process is determinantal with correlation kernel

$$K_n(z, w) = \frac{1}{n\Delta Q(p_n)} \mathbf{K}_n(\zeta, \eta),$$

where

$$z = e^{-i\theta_n} \sqrt{n\Delta Q(p_n)} (\zeta - p_n), \quad w = e^{-i\theta_n} \sqrt{n\Delta Q(p_n)} (\eta - p_n).$$

We write $R_n(z) = K_n(z, z)$.

Lemma 1.1. *There is a sequence of cocycles c_n such that every subsequence of $c_n K_n$ has a subsequence converging to $G\Psi$ where*

$$G(z, w) = e^{z\bar{w} - |z|^2/2 - |w|^2/2}$$

is the Ginibre kernel and Ψ is a Hermitian entire function satisfying the mass-one inequality

$$(1.4) \quad \int_{\mathbb{C}} e^{-|z-w|^2} |\Psi(z, w)|^2 dA(w) \leq \Psi(z, z).$$

Further, the function $R(z) := K(z, z) = \Psi(z, z)$ is either identically zero, or else it is everywhere strictly positive.

Proof. The first statement follows from a normal families argument; see Theorem A in [3]. The last statement is proven in [3], Theorem B. \square

A limit point K in Lemma 1.1 will be called a *limiting kernel*, and

$$R(z) := K(z, z)$$

is the corresponding *limiting 1-point function*. The general theory of point fields implies that a limiting kernel K is the correlation kernel of a "limiting infinite point field" Θ with k -point function $R_k(z_1, \dots, z_k) = \det(K(z_i, z_j))_{i,j=1}^k$. See [3] and the references there.

There is nothing which prevents the limiting kernel K from being trivial, i.e., we may well have $K = 0$. In this case the point process Θ_n degenerates to the trivial point field, all of whose k -point functions vanishes identically. As we shall see, this happens when we rescale about certain types of singularities. However, when we shift focus and rescale about a suitable moving point, approaching the singular point from the inside of the droplet at a suitable distance away, we will obtain new, non-degenerate, limiting point fields.

If R is non-trivial, we can go on to define the Berezin kernel

$$B(z, w) = \frac{|K(z, w)|^2}{R(z)}$$

and its Cauchy transform

$$C(z) = \int_{\mathbb{C}} \frac{B(z, w)}{z - w} dA(w).$$

By Theorem B in [3], the function C is smooth and we have Ward's equation

$$(1.5) \quad \bar{\partial}C = R - 1 - \Delta \log R.$$

Since K (and therefore B and C) is uniquely determined by R by polarization, it makes sense to say that R *satisfies Ward's equation* if (1.5) holds.

1.3. Main results. Let p be a singular boundary point, i.e., a cusp or a double point.

Theorem I. *Let $e^{i\theta}$ be one of the normal directions to ∂S at p and rescale about p according to*

$$z_j = e^{-i\theta} \sqrt{n\Delta Q(p)}(\zeta_j - p), \quad j = 1, \dots, n.$$

Then any limiting 1-point function R vanishes identically. As a consequence, any limiting point field at p is trivial.

Let us remark that the choice of angle θ above is merely made for convenience. Any other angle, or sequence θ_n of angles, would produce the same result.

Since Theorem I says that nothing of interest is to be found in the vicinity of the singular point p , we shift focus and look a bit to the inside of the droplet. We need to consider the case of a cusp and a double point separately.

Definition. Fix a positive parameter T .

- (i) If ∂S has a cusp at p , we consider the moving point $p_n \in S$ of distance $T/\sqrt{n\Delta Q(p)}$ from the boundary ∂S , which is closest to the singular point p . (See Fig. 1.)
- (ii) If S has a double point, there are instead two distinct points p'_n, p''_n in S of distance $T/\sqrt{n\Delta Q(p)}$ to ∂S , of minimal distance to p .

Theorem II. *Suppose that S has a $(\nu, 2)$ -cusp at p where $\nu \geq 5$. Rescale about p_n according to*

$$(1.6) \quad z_j = e^{-i\theta_n} \sqrt{n\Delta Q(p_n)}(\zeta_j - p_n), \quad j = 1, \dots, n,$$

where the angle θ_n is chosen so that the image of the cusp, i.e. the point $e^{-i\theta_n}(p - p_n)$, is on the positive imaginary axis. Then, if T is sufficiently large, each limiting 1-point function $R(z) = K(z, z)$ is everywhere positive and satisfies Ward's equation. Moreover, R satisfies the estimate

$$(1.7) \quad R(z) \leq Ce^{-2(|x|-T)^2}, \quad (x = \operatorname{Re} z).$$

The assumption that the parameter $T > 0$ be sufficiently large is made for technical reasons of the proof; it should really not be needed. See the concluding remarks for related comments.

Our result for double points is similar.

Theorem III. *If S has a double point at p , we rescale as in (1.6) with p_n replaced by p'_n or p''_n . The conclusions of Theorem II then hold also for the limiting 1-point function R rescaled about p'_n or p''_n .*

We remark that the limiting point fields, whose existence is guaranteed by Theorems II and III are necessarily different from those which can appear at a fixed regular boundary point. Indeed, as was observed in [3], a limiting 1-point function rescaled in the outer normal direction about a regular boundary point will satisfy the estimate

$$|R(z) - \chi_{(-\infty, 0)}(x)| \leq Ce^{-cx^2}$$

where c is some positive constant. The latter estimate is not consistent with (1.7), which is seen by letting $x \rightarrow -\infty$.

The concluding remarks in Section 5 contains further comments about the limiting point fields in theorems II and III.

1.4. Organization of the paper. Our approach uses Bergman space techniques and some rather careful estimates for the limiting 1-point function in the *exterior* of the droplet. It is convenient to start with the latter estimates.

In Section 2, we prove an estimate for the decay of a limiting 1-point function in the exterior of the droplet, near a cusp. In Section 3, we extend the estimate to rather general (moving) boundary points. In Section 4 we prove our main results, Theorems I, II, and III. Section 5 contains concluding remarks.

2. EXTERIOR ESTIMATE FOR THE 1-POINT FUNCTION AT A CUSP

In this section, we assume that the droplet S has a cusp at the point $p \in \partial S$. To simplify the discussion, we will assume that p on the *outer* boundary of S .

Now let $e^{i\theta}$ be one of the normal directions to ∂S and rescale about p in the usual manner:

$$z_j = e^{-i\theta} \sqrt{n\Delta Q(p)}(\zeta_j - p), \quad j = 1, \dots, n.$$

Let $K = G\Psi$ denote a limiting kernel. Write $R(z) = K(z, z)$ and $R_n(z) = K_n(z, z)$. We have the following result.

Theorem 2.1. *With the above assumptions, we have*

$$(2.1) \quad R(z) \leq Ce^{-2x^2}, \quad (x = \operatorname{Re} z).$$

To prove the estimate (2.1) we shall use the well-known estimate (e.g. [3], Section 3)

$$(2.2) \quad \mathbf{R}_n(\zeta) \leq C n e^{-n(Q-\check{Q})(\zeta)}.$$

Here \check{Q} is the "obstacle function", i.e. the $C^{1,1}$ -smooth function on \mathbb{C} which coincides with Q on S and is harmonic in $\mathbb{C} \setminus S$ and increases like $\check{Q}(\zeta) = \log |\zeta|^2 + O(1)$ as $\zeta \rightarrow \infty$.

Now assume *w.l.o.g.* that $p = 0$ and that the cusp at p points in the positive real direction. The rescaling is then simply given by

$$(2.3) \quad z = i\sqrt{n\Delta Q(p)} \zeta,$$

and in the z -plane, the droplet appears as a narrow neighbourhood of the ray $(-\infty, 0)$.

Write U for the unbounded component of $\mathbb{C} \setminus S$ and let $\Phi : \mathbb{C}_+ \rightarrow U$ be a conformal map such that $\Phi(0) = 0$ and $\Phi(i) = \infty$. Since 0 is a conformal cusp, Φ extends analytically to some neighbourhood N of the origin. Likewise, the harmonic function $\check{Q} \circ \Phi : \mathbb{C}_+ \rightarrow \mathbb{R}$ extends harmonically across \mathbb{R} to a harmonic function V . Referring to (2.3), we shall put

$$\zeta = \Phi(\lambda).$$

We can assume that Φ' has the Taylor expansion

$$\Phi'(\lambda) = \lambda + a_2 \lambda^2 + a_3 \lambda^3 \dots, \quad (\lambda \in N).$$

We form the functions

$$Q_\Phi := Q \circ \Phi, \quad \check{Q}_\Phi := \check{Q} \circ \Phi.$$

The function \check{Q}_Φ is harmonic in \mathbb{C}_+ and extends across \mathbb{R} to a harmonic function V . Write

$$M(\lambda) := (Q_\Phi - V)(\lambda), \quad \lambda = \sigma + i\tau.$$

Thus $M = (Q - \check{Q}) \circ \Phi$ in \mathbb{C}_+ .

Lemma 2.2. *For $\lambda = \sigma + i\tau$, we have*

$$(2.4) \quad M(\lambda) = 2\Delta Q(p)\tau^2\sigma^2 + O(|\lambda|^5), \quad (\lambda \rightarrow 0).$$

Before we prove the lemma, we proceed to prove the estimate (2.1). To this end, note that the estimate (2.2) gives (with a new C depending on $\Delta Q(p)$)

$$(2.5) \quad R_n(z) \leq C e^{-nM(\lambda_n(z))}, \quad \text{where } \lambda_n(z) := \Phi^{-1}\left(-iz/\sqrt{n\Delta Q(p)}\right).$$

If $z = x + iy$, then, since $\Phi(\lambda) = \lambda^2/2 + O(\lambda^3)$ as $\lambda \rightarrow 0$,

$$(2.6) \quad -x = \sqrt{n\Delta Q(p)} \operatorname{Im}(\lambda^2/2 + O(\lambda^3)) = \sqrt{n\Delta Q(p)}(\sigma\tau + O(|\lambda|^3)), \quad (\lambda = \sigma + i\tau \rightarrow 0).$$

The estimates (2.4) and (2.6) now give that

$$nM(\lambda_n(z)) = 2x^2 + O(n|\lambda_n(z)|^5), \quad (n \rightarrow \infty).$$

Choosing, for example, $|z| \leq \log n$, we see via (2.5) that the estimate (2.1) holds.

It remains to prove Lemma 2.2.

Recalling that $\check{Q}(\lambda) \sim \log |\lambda|^2$ as $\lambda \rightarrow \infty$, it is now seen that the Poisson representation of the harmonic function $\check{Q}_\Phi|_{\mathbb{C}_+}$ takes the form

$$(2.7) \quad \check{Q}_\Phi(\lambda) := \int_{\mathbb{R}} Q_\Phi(t) P(\lambda, t) dt + G(\lambda), \quad \lambda \in \mathbb{C}_+,$$

where $G(\lambda) = \log \left| \frac{\lambda+i}{\lambda-i} \right|^2$ is (twice) the Green's function for \mathbb{C}_+ with pole at i , and

$$P(\lambda, t) = \frac{1}{\pi} \frac{\tau}{(\sigma - t)^2 + \tau^2}, \quad \lambda = \sigma + i\tau$$

the Poisson kernel for \mathbb{C}_+ .

Let us write C^ω for the real-analytic class. For a function $f \in C^\omega(\mathbb{R})$ we write $P_f(\lambda) = \int_{\mathbb{R}} f(t) P(\lambda, t) dt$ for the Poisson integral. Noting that, for $\lambda = \sigma + i\tau \in \mathbb{C}_+$,

$$(\sigma - t)^2 P(\lambda, t) - \tau/\pi = -\tau^2 P(\lambda, t),$$

we compute

$$\begin{aligned} (2.8) \quad P_f(\lambda) - f(\sigma) &= \int_{\mathbb{R}} [f(t) - f(\sigma) - (t - \sigma)f'(\sigma)] P(\lambda, t) dt \\ &= \int_{\mathbb{R}} \left[\frac{f(t) - f(\sigma) - (t - \sigma)f'(\sigma)}{(\sigma - t)^2} \right] (\sigma - t)^2 P(\lambda, t) dt \\ &= \frac{\tau}{\pi} \cdot \text{p. v.} \int_{\mathbb{R}} \frac{f(t) - f(\sigma) - (t - \sigma)f'(\sigma)}{(\sigma - t)^2} dt - \tau^2 \int_{\mathbb{R}} \frac{f(t) - f(\sigma) - (t - \sigma)f'(\sigma)}{(\sigma - t)^2} P(\lambda, t) dt. \end{aligned}$$

Note that the last integral approaches $\frac{1}{2}f''(\sigma)$ as $\tau \downarrow 0$.

Let us denote by $S_\sigma : C^\omega(\mathbb{R}) \rightarrow C^\omega(\mathbb{R})$ the backward shift by σ : $S_\sigma f(t) = \frac{f(t) - f(\sigma)}{t - \sigma}$. Write

$$I_f^k(\sigma) := \text{p. v.} \int_{\mathbb{R}} S_\sigma^{2k} f(t) dt, \quad k = 1, 2, \dots$$

A repetition of the calculation in (2.8) gives that P_f has the asymptotic expansion

$$(2.9) \quad P_f(\sigma + i\tau) = f(\sigma) + I_f^1(\sigma) \cdot \tau - \frac{1}{2}f''(\sigma) \cdot \tau^2 - I_f^2(\sigma) \cdot \tau^3 + \frac{1}{4!}f^{(4)}(\sigma) \cdot \tau^4 + \dots$$

Choosing $f = Q_\Phi$ and using identity (2.7), we find

$$I_{Q_\Phi}^1(\sigma) = \partial_\tau P_{Q_\Phi}(\sigma) = \partial_\tau(Q_\Phi - G)(\sigma), \quad \sigma \in \mathbb{R}.$$

More generally, it is easy to verify by induction that

$$(2.10) \quad (2k - 1)! \cdot I_{Q_\Phi}^k(\sigma) = \partial_\sigma^{2k-2} \partial_\tau(Q_\Phi - G)(\sigma).$$

Since $\partial \check{Q}_\Phi$ is continuous on $\text{cl } \mathbb{C}_+$, while $Q_\Phi = \check{Q}_\Phi$ in the lower half plane, we can replace " Q_Φ " by " \check{Q}_Φ " in the right side of (2.10), and $M = Q_\Phi - P_{Q_\Phi} - G$ satisfies $M = \partial_\tau M = 0$ on \mathbb{R} . Inserting the expansions (2.9) and (2.10), we thus find that

$$\begin{aligned} M(\sigma + i\tau) &= \frac{1}{2} \partial_\tau^2 M(\sigma) \cdot \tau^2 + \frac{1}{3!} \partial_\tau^3 M(\sigma) \cdot \tau^3 + \frac{1}{4!} \partial_\tau^4 M(\sigma) \cdot \tau^4 + \dots \\ &= \frac{1}{2} \partial_\tau^2 Q_\Phi(\sigma) \cdot \tau^2 + \frac{1}{3!} \partial_\tau^3 Q_\Phi(\sigma) \cdot \tau^3 + \frac{1}{4!} \partial_\tau^4 Q_\Phi(\sigma) \cdot \tau^4 + \dots \\ &\quad + \frac{1}{2} \partial_\sigma^2 Q_\Phi(\sigma) \cdot \tau^2 + \frac{1}{3!} \partial_\sigma^2 \partial_\tau Q_\Phi(\sigma) \cdot \tau^3 - \frac{1}{4!} \partial_\sigma^4 Q_\Phi(\sigma) \cdot \tau^4 + \dots \\ &\quad - \frac{1}{2} \partial_\tau^2 G(\sigma) \cdot \tau^2 - \frac{1}{3!} \partial_\tau (\partial_\tau^2 + \partial_\sigma^2) G(\sigma) \cdot \tau^3 - \frac{1}{4!} \partial_\tau^4 G(\sigma) \cdot \tau^4 + \dots \end{aligned}$$

The last line vanishes because all coefficients are derivatives of ΔG evaluated at σ . This implies that

$$(2.11) \quad M(\sigma + i\tau) = 2\Delta Q_\Phi(\sigma) \cdot \tau^2 + \frac{4}{3!} \partial_\tau \Delta Q_\Phi(\sigma) \cdot \tau^3 + \frac{4}{4!} (\partial_\tau^2 - \partial_\sigma^2) \Delta Q_\Phi(\sigma) \cdot \tau^4 + \dots$$

But $\Delta Q_\Phi(\sigma + i\tau) = \Delta Q(\Phi(\sigma + i\tau)) |\Phi'(\sigma + i\tau)|^2 = \Delta Q(\Phi(\sigma + i\tau)) \cdot (\sigma^2 + \tau^2 + \dots)$. We have shown that

$$M(\sigma + i\tau) \geq 2\Delta Q(0) \cdot \sigma^2 \tau^2 + O(|\lambda|^5), \quad (\lambda = \sigma + i\tau \rightarrow 0).$$

The proof of Lemma 2.2 is complete. \square

3. A GENERAL EXTERIOR ESTIMATE FOR THE 1-POINT FUNCTION

In this section, we consider a (possibly moving) boundary point p and rescale about p in the outer normal direction $e^{i\theta}$ in the usual way,

$$z_j = e^{-i\theta} \sqrt{n\Delta Q(p)} (\zeta_j - p), \quad j = 1, \dots, n.$$

We can then form the rescaled 1-point function $R_{n,p}(z)$, and also sequential limiting 1-point functions

$$R_p(z) = \lim_{k \rightarrow \infty} R_{n_k,p}(z), \quad z \in \mathbb{C}.$$

In the situations relevant to this paper, we find that the decay of $R_p(z)$ is at least like e^{-2x^2} where $x = \operatorname{Re} z$. Our estimates are however not quite uniform; there is a critical case of boundary points at distance about $n^{-1/3}$ to the closest singular boundary point when our method is inconclusive. Since such points play no role to our analysis anyway, we will simply disregard them here.

Theorem 3.1. *Let δ_n be a sequence of positive numbers with $\delta_n \rightarrow 0$ and $n^{1/3}\delta_n \rightarrow \infty$ as $n \rightarrow \infty$. Then there is a constant C such that if the distance from a boundary point p to the nearest singular point is at least δ_n , then $R_{n,p}(z) \leq Ce^{-2x^2}$, when $|z| \leq \log n$, $x = \operatorname{Re} z$. As a consequence, any limiting 1-point function R_p satisfies the estimate*

$$R_p(z) \leq Ce^{-2x^2}.$$

As a simple corollary, we will obtain the estimate (1.7) in Theorem II.

The proof will be carried out in steps, in the following three paragraphs. We will first treat the simple case of an "ordinary" fixed boundary point p . After that we consider cases of moving points near singular boundary points.

3.1. Fixed boundary points. Let p be an arbitrary fixed boundary point of S such that $\Delta Q(p) > 0$. We rescale about p according to

$$z = \sqrt{n\Delta Q(p)} e^{-i\theta} (\zeta - p)$$

where $e^{i\theta}$ is chosen as an outer normal direction to ∂S . The case when p is a cusp was already treated in Theorem 2.1. Double points can be treated similarly to regular points, so we give complete details only in the regular case.

Thus suppose that p is a regular boundary point at distance at least δ from all singular boundary points, where $\delta > 0$ is independent of n . It will suffice to prove that there is a constant $C = C(\delta)$ such that

$$(3.1) \quad \zeta \in S^c, \quad |z| \leq \log n \quad \Rightarrow \quad R_n(z) \leq Ce^{-2x^2}, \quad (z = x + iy).$$

As before, we shall use the estimate $\mathbf{R}_n(\zeta) \leq Cne^{-n(Q-\tilde{Q})(\zeta)}$. Let N_p be the outer normal direction at p and let V be the harmonic continuation of $\tilde{Q}|_{S^c}$ to a neighbourhood of p . We write $\delta_n = C \log n / \sqrt{n}$.

By Taylor's formula we have, for $x > 0$, with $M := Q - V$,

$$M\left(p + \frac{x}{\sqrt{n\Delta Q(p)}}N_p\right) = \frac{1}{2n\Delta Q(p)} \frac{\partial^2 M}{\partial n^2}(p) x^2 + \frac{1}{6(n\Delta Q(p))^{3/2}} \frac{\partial^3 M}{\partial n^3}(p + \theta)x^3,$$

where " $\partial/\partial n$ " is exterior normal derivative and $\theta = \theta(n, p)$ is some number between 0 and δ_n . However, since p is a regular point and $Q = V$ on ∂S we have $\frac{\partial^2 M}{\partial s^2}(p) = 0$ where $\partial/\partial s$ denotes differentiation in the tangential direction. Adding this to the above Taylor expansion, using that $(\partial_s^2 + \partial_n^2)M = 4\Delta M = 4\Delta Q$, we obtain, when $|z| \leq C \log n$,

$$(3.2) \quad nM\left(p + \frac{z}{\sqrt{n\Delta Q(p)}}N_p\right) = 2x^2 + O\left(\frac{\log^3 n}{\sqrt{n}}\right), \quad 0 \leq x \leq \delta_n.$$

This proves the desired implication (3.1).

3.2. Boundary points near cusps. Assume now that the droplet S has a $(\nu, 2)$ -cusp at the origin, pointing in the positive real direction. Here $\nu \geq 5$ is an odd integer.

Fix $T > 0$ and a large integer n . Let p_n be the unique point such that the disk

$$D_n := D\left(p_n; T/\sqrt{n\Delta Q(0)}\right)$$

is inside S and is tangent to ∂S at two points; one on each arc terminating in the cusp. Let q_n be the point in $(\partial D_n) \cap (\partial S)$ in the upper half-plane. It is easy to check that $|p_n| \sim n^{-1/\nu}$ and $|q_n| \sim n^{-1/\nu}$. (Here " $a_n \sim b_n$ " means that there is a constant $C > 0$ such that $C^{-1}a_n \leq b_n \leq Ca_n$.)

We rescale about q_n as follows. Let $e^{i\theta_n}$ be the outer normal to ∂S at q_n and put

$$(3.3) \quad z = e^{-i\theta_n} \sqrt{n\Delta Q(0)} (\zeta - q_n).$$

The rescaled droplet near q_n looks roughly like the infinite strip

$$0 < \operatorname{Re} z < 2T,$$

and the image of the cusp is far away on the positive imaginary axis at a distance $\sim n^{1/2-1/\nu}$ from the origin.

Let Φ be a conformal map $\mathbb{C}_+ \rightarrow U$ where \mathbb{C}_+ is the upper half-plane and U is the complement of the component of S containing 0. We assume that $\Phi(0) = 0$ and

$$\Phi(\lambda) = -\frac{1}{2}\lambda^2 + O(\lambda^3), \quad \lambda \rightarrow 0.$$

Since the cusp is conformal, Φ extends analytically across \mathbb{R} .

Now let σ_n be the point on \mathbb{R} such that $\Phi(\sigma_n) = q_n$. We will be dealing with the Taylor expansion about σ_n given by

$$\Phi(\sigma_n + \tilde{\lambda}) = q_n + \Phi'(\sigma_n)\tilde{\lambda} + \Phi''(\sigma_n)\frac{\tilde{\lambda}^2}{2} + \cdots.$$

We now note the simple approximation

$$|\Phi'(\sigma_n)| \sim |\sigma_n| \sim |q_n|^{1/2} \sim n^{-1/2\nu},$$

which implies that the condition $|z| \leq \log n$ in (3.3) corresponds to

$$(3.4) \quad |\tilde{\lambda}| \leq C_0 \frac{\log n}{n^{1/2-1/(2\nu)}}.$$

Observe first that it follows from (3.4) that

$$(3.5) \quad n|\tilde{\lambda}|^3 \leq C_0 n \frac{\log^3 n}{n^{3/2-3/(2\nu)}} \leq C_0 \frac{\log^3 n}{n^{(\nu-3)/(2\nu)}} \rightarrow 0, \quad (n \rightarrow \infty)$$

where the last assertion holds true since $\nu \geq 5$.

After these observations, we can prove the required decay about the moving point q_n .

Lemma 3.2. *Let $R_n(z)$ be the rescaled one-point function according to the rescaling (3.3). There is then a constant C such that $R_n(z) \leq Ce^{-2x^2}$ when $|z| \leq \log n$ and $x = \operatorname{Re} z < 0$.*

Proof. We know that $\mathbf{R}_n(\zeta) \leq Cne^{-n(Q-\tilde{Q})(\zeta)}$. We will put $\zeta = \sigma_n + \tilde{\lambda}$ and compare $2x^2$ with $n(Q-\tilde{Q})(\zeta)$. We shall be done if we can prove that, for $x > 0$ we have

$$(3.6) \quad 2x^2 = n(Q-\tilde{Q})(\sigma_n + \tilde{\lambda}) + o(1), \quad |\tilde{\lambda}| \leq C_0 \frac{\log n}{n^{1/2-1/(2\nu)}}.$$

However, by the estimate (2.11) we have

$$(3.7) \quad n(Q-\tilde{Q})(\sigma_n + \tilde{\lambda}) = 2\Delta Q(q_n) |\Phi'(\sigma_n)|^2 \tau^2 + O(n|\tilde{\lambda}|^3), \quad (\tilde{\lambda} = \sigma + i\tau).$$

Inserting here the Taylor expansion of Φ about σ_n we find that

$$z = i\sqrt{n\Delta Q(0)} \left(|\Phi'(\sigma_n)| \tilde{\lambda} + \alpha_n \frac{\tilde{\lambda}^2}{2} + \dots \right),$$

where $|\alpha_n| = |\Phi''(\sigma_n)|$ is $O(1)$. It follows that

$$x^2 = n\Delta Q(0) \left(|\Phi'(\sigma_n)| \tau + O(\tilde{\lambda}^2) \right)^2 = n\Delta Q(0) \left(|\Phi'(\sigma_n)|^2 \tau^2 + O(\tilde{\lambda}^3) \right),$$

which leads to

$$\begin{aligned} 2x^2 - n(Q-\tilde{Q})(q_n + w/\sqrt{n}) &= 2n\Delta Q(0) \left(|\Phi'(\sigma_n)|^2 \tau^2 + O(\tilde{\lambda}^3) \right) - 2n\Delta Q(0) \left(|\Phi'(\sigma_n)|^2 \tau^2 + O(\tilde{\lambda}^3) \right) \\ &= O(n|\tilde{\lambda}|^3) \rightarrow 0, \quad (n \rightarrow \infty) \end{aligned}$$

by (3.5). The proof of the lemma is finished. \square

Let us now change the rescaling so that p_n is mapped to the origin,

$$z = e^{-i\theta_n} \sqrt{n\Delta Q(0)} (\zeta - p_n),$$

and write R_n for the rescaled 1-point function. Since

$$|p_n - q_n| = \operatorname{Re}(q_n + p_n) + o(1) = T/\sqrt{n\Delta Q(0)}$$

we obtain the estimate

$$(3.8) \quad R_n(z) \leq Ce^{-2(|x|-T)^2}, \quad |z| \leq \log n, \quad x = \operatorname{Re} z.$$

3.3. Boundary points near double points. The case of double points is now quite similar to the case of cusps. Assume that the droplet has a double point at the origin such that the boundary is there tangent to the real axis. If we fix $t > 0$ there are two points p'_n, p''_n in S whose distance to 0 is minimal, and such that the distance to the boundary equals $T/\sqrt{n\Delta Q(0)}$. The important observation is that the distance from these points to the origin is $\sim n^{-1/\nu}$ where $\nu \geq 4$. Indeed, if the two arcs meeting at the double point has first-order tangency, then a simple geometric consideration shows that $\nu = 4$. Otherwise, the two arcs have higher-order tangency, and then the points p'_n, p''_n are even further from the origin, i.e. $\nu > 4$.

We can now perform the exact same analysis as in Section 3.2 with p_n replaced by one of the points p'_n or p''_n . The crucial observation is that the estimate (3.5) persists, since $4 > 3$.

We have now completely proved Theorem 3.1. Since the estimate (3.8) holds also for the case of double points, we obtain as a corollary the estimate (1.7) in Theorem II. q.e.d.

4. PROOFS OF THE MAIN RESULTS

4.1. The triviality theorem. We now prove Theorem I. Suppose that p is either a double point or a cusp of type $(\nu, 2)$ where $\nu > 3$ and that $\Delta Q(p) > 0$ and rescale about p according to

$$z = e^{-i\theta} \sqrt{n\Delta Q(p)}(\zeta - p)$$

where $e^{i\theta}$ is one of the normal directions to ∂S at p .

Let $K = G\Psi$ be a limiting kernel. We must prove that the limiting 1-point function $R(z) = K(z, z) = \Psi(z, z)$ vanishes identically. To this end, we shall use the corresponding *holomorphic kernel*

$$L(z, w) = e^{z\bar{w}} \Psi(z, w).$$

Lemma 4.1. *The function $z \mapsto L(z, z)$ is logarithmically subharmonic.*

Proof. We first note that K is a *positive matrix*, i.e. $\sum_{j,k=1}^N \alpha_j \bar{\alpha}_k K(z_j, z_k) \geq 0$ for all choices of finitely many scalars α_j and all points z_j . The positivity is clear from the fact that

$$\det(K(z_j, z_k))_{j,k=1}^N = R_N(z_1, \dots, z_N) \geq 0$$

for all finite N and all $(z_j)_1^N \in \mathbb{C}^N$. Now, since

$$L(z, w) = e^{|z|^2/2} e^{|w|^2/2} K(z, w)$$

it follows that L is a positive matrix as well. Moreover, the function L is Hermitian entire.

Following Aronszajn [5], we then consider the semi-normed Hilbert space \mathcal{H}_* which is the completion of finite linear combinations $\sum_1^N \alpha_j L_{z_j}$ under the inner product

$$\langle L_z, L_w \rangle_* := L(w, z), \quad (L_z(w) = L(w, z)).$$

The function

$$F(z, w) = \langle L_w, L_z \rangle_*$$

is Hermitian entire, so we have $\partial_z F(z, w) = \langle L_w, \bar{\partial}_z L_z \rangle_*$, $\bar{\partial}_w \partial_z F(z, w) = \langle \bar{\partial}_w L_w, \bar{\partial}_z L_z \rangle_*$, etc. It follows that if $L(z, z) > 0$ then

$$\Delta_z \log L(z, z) = \frac{\|\bar{\partial}_z L_z\|_*^2 \cdot \|L_z\|_*^2 - |\langle \bar{\partial}_z L_z, L_z \rangle_*|^2}{L(z, z)^2} \geq 0$$

by the Cauchy-Schwarz inequality. Since $\log L(z, z) = -\infty$ when $L(z, z) = 0$, $\log L(z, z)$ has the sub-mean value property on \mathbb{C} . Since the function is also lower semicontinuous, it is subharmonic. The proof of the lemma is complete. \square

We can now finish the proof of Theorem I.

By Lemma 4.1, the function $S(z) := |z|^2 + \log R(z)$ is subharmonic. Now recall the estimate $R(z) \leq Ce^{-2x^2}$. (See Theorem 2.1 for cusps and §3.1 for double points.) This gives

$$S(z) \leq \log C + y^2 - x^2.$$

But $y^2 - x^2$ is harmonic, so the function $\tilde{S} = S - (y^2 - x^2)$ is subharmonic and bounded above by $\log C$. Hence it is constant, i.e.,

$$R(z) = Ce^{-2x^2}$$

for a (new) constant C . If R is nontrivial we can assume that $C = 1$. By polarization, then

$$\Psi(z, w) = e^{-(z+\bar{w})^2/2},$$

so the kernel $L(z, w) = e^{z\bar{w}}\Psi(z, w)$ must satisfy

$$\int |L(0, w)|^2 e^{-|w|^2} dA(w) = \int |\Psi(0, w)|^2 e^{-|w|^2} dA(w) = \int e^{-2x^2} dA = \infty.$$

This contradicts Lemma 4.9 in [3]. The contradiction shows that we must have $C = 0$. q.e.d.

We are grateful to Håkan Hedenmalm for helpful communication in connection with the above proof, [11].

4.2. Proof of the existence theorems. We now prove the existence theorems, Theorem II and III.

Let p be either a $(\nu, 2)$ -cusp with $\nu \geq 5$ or a double point. In both cases we assume that $\Delta Q(p) > 0$. Also fix a number $T > 0$. For a given $n \in \mathbb{Z}_+$, we let p_n be a point in S whose distance to the boundary is $T/\sqrt{n\Delta Q(p)}$ and whose distance to p is minimal.

We rescale about p_n ,

$$z_j = e^{-i\theta_n} \sqrt{n\Delta Q(p)} (\zeta_j - p_n), \quad j = 1, \dots, n,$$

where the angle θ_n is chosen so that the image of the point p lies on the positive imaginary axis.

Note that as $n \rightarrow \infty$, the image of S near p_n looks approximately like the strip

$$(4.1) \quad \Sigma_T : -T < \operatorname{Re} z < T.$$

Let K_n be the kernel of the rescaled system $\Theta_n = \{z_j\}_1^n$. We write $R_n(z) = K_n(z, z)$. By Lemma 1.1 we know that there is a sequence of cocycles c_n such that every subsequence of $c_n K_n$ has a subsequence converging to $G\Psi$ where Ψ is some Hermitian entire function. It remains only to show that the function $R(z) = \Psi(z, z)$ does not vanish identically if T is large enough.

To this end, we shall use the estimate in [3], Theorem 5.4,

$$|\mathbf{R}_n(\zeta) - n\Delta Q(\zeta)| \leq C \left(1 + ne^{-n\ell\Delta Q(\zeta) \cdot \delta(\zeta)^2}\right), \quad \zeta \in S,$$

where ℓ is a positive constant and $\delta(\zeta) = \operatorname{dist}(\zeta, \partial S)$. If we choose $\zeta = p_n$ where $\delta(p_n) = T/\sqrt{n\Delta Q(p)}$, we obtain for the rescaled 1-point function R_n that

$$|R_n(0) - 1| \leq Ce^{-\ell T^2}.$$

choosing T sufficiently large that the right hand side is < 1 , we obtain that $R(0) > 0$. An application on Lemma 1.1 now shows that $R(z) > 0$ for all $z \in \mathbb{C}$. q.e.d.

5. CONCLUDING REMARKS

We shall here discuss a family of natural candidates of the limiting point fields whose existence is guaranteed by theorems II and III.

Let p be a singular point; for definiteness, let us say it is a cusp.

For a "large" T , consider the point $p_n \in S$ at distance $T/\sqrt{n\Delta Q(p)}$ from ∂S being closest to p and rescale about p_n as in Theorem II. Let $K = G\Psi$ be a limiting kernel in Lemma 1.1; then K is non-trivial.

Recall that a Hermitian-entire function Ψ is called translation invariant (in short: *t.i.*) if $\Psi(z, w) = \Phi(z + \bar{w})$ for some entire function Φ . We do not know that a limiting kernel must be *t.i.* but it seems to be a reasonable assumption, since the rescaled droplet looks like the strip

$$\Sigma_T = \{z; -T \leq \operatorname{Re} z \leq T\}.$$

In any case, we shall now use theory from [3] to narrow down the set of possible limiting kernels, under the extra hypothesis of translation invariance.

First of all, Theorem E in [3] implies that a *t.i.* limiting kernel $K(z, w) = G(z, w)\Phi(z + \bar{w})$ has the "Gaussian representation"

$$\Phi(z) = \gamma * f(z) = \int_{-\infty}^{+\infty} \gamma(z - t)f(t) dt,$$

where $\gamma(z) = (2\pi)^{-1/2}e^{-z^2/2}$ is the Gaussian kernel and $f(t)$ is some Borel function with $0 \leq f \leq 1$. Secondly, Theorem F says that the function Φ above gives rise to a solution to Ward's equation (1.5) if and only if $f = \chi_I$ is the characteristic function of some *interval* I .

Finally, by the estimate (1.7), we know that the function $R(z) = \Phi(z + \bar{z})$ must satisfy

$$\Phi(2x) \leq Ce^{-2(|x|-T)^2}, \quad x \in \mathbb{R}.$$

In order for this to be consistent with the identity

$$\Phi(x) = \gamma * \chi_I(x) = \frac{1}{\sqrt{2\pi}} \int_I e^{-(x-t)^2/2} dt,$$

we must have $I \subset [-2T, 2T]$.

Thus if there exists a limiting *t.i.* kernel, it necessarily has the structure $\Phi = \gamma * \chi_I$ where I is an interval contained in $[-2T, 2T]$.

For reasons on symmetry, it is natural to assume that an interval I as above can be chosen to be symmetric, i.e. $I = [-s/2, s/2]$ for some number $s = S(T)$ between 0 and $4T$.

Let us denote by Φ_s the function

$$\Phi_s(z) := \gamma * \chi_{(-s/2, s/2)}(z) = \frac{1}{\sqrt{2\pi}} \int_{-s/2}^{s/2} e^{-(z-t)^2/2} dt.$$

We know from Lemma 7.7 in [3] that the kernel $K_s(z, w) := G(z, w)\Phi_s(z + \bar{w})$ appears as the correlation kernel of a unique point field.

It is close at hand to guess that one of the K_s ($0 < s \leq 4T$) will appear as a limiting kernel in Theorem II.

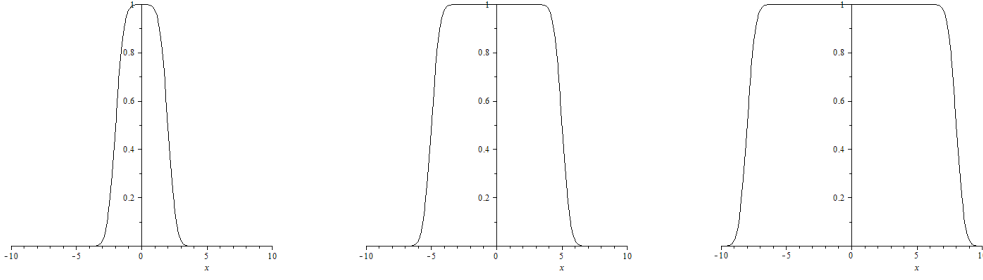


FIGURE 3. The graph of $R_s(x) := \gamma * \chi_{(-s/2, s/2)}(2x)$ for $s = 2$, $s = 5$, and $s = 8$.

To further corroborate our point, we may observe that the kernel K_s interpolates in a natural way between the trivial kernel $K_0 = 0$ (at the singular point) and the Ginibre kernel $K_\infty = G$ (the bulk regime). See Figure 3.

Now fix a large enough number T . The conditions shown for a limiting 1-point function R , i.e., that it be non-trivial, satisfy Ward's equation, and have the decay $R(x) \leq Ce^{-2(|x|-T)^2}$, are not enough to fix a solution uniquely, even under the assumption of translation invariance.

In the paper [3], the corresponding question was solved for regular boundary points, by using the 1/8-formula ([3], Theorem D)

$$(5.1) \quad \int_{\mathbb{R}} t \cdot (\chi_{(-\infty, 0)}(t) - R(t)) dt = \frac{1}{8}.$$

This suggests that, in order to single out a unique solution, one could seek some additional condition similar to (5.1), which is to hold for a limiting kernel near a singular boundary point. One of the main challenges with this is that the boundary fluctuation theorem from [2], which was used to prove the condition (5.1), is so far only known for connected droplets with everywhere smooth boundary. Thus, it would seem natural to try to extend the methods in [2] to domains with singular boundary points.

We also mention that the reason that we had to insist that the parameter T be sufficiently large is that we have used the technique of Hörmander estimates to ensure non-triviality of the 1-point function, and Hörmander estimates require a certain distance to the boundary. On the other hand, if one could prove a generalized boundary fluctuation theorem, as suggested above, one would probably obtain non-triviality also for small $T > 0$.

There is a parallel "hard-edge" theory, where one confines the system $\{\zeta_j\}_1^n$ to the droplet by redefining Q outside the droplet to be $+\infty$ there. Similar results to those studied in this paper (the "free boundary" case) can be obtained; cf. [3] and [4]. In the following we will freely use results from the forthcoming paper [4].

If one rescales near a singular boundary point p in the hard-edge case, at distance $T/\sqrt{n\Delta Q(p)}$ from the boundary, one obtains limiting kernels of the form

$$K(z, w) = G(z, w) \Psi(z, w) \chi_{(-T, T)}(\operatorname{Re} z) \chi_{(-T, T)}(\operatorname{Re} w)$$

where Ψ is some Hermitian-entire function. In the *t.i.* case $\Psi(z, w) = \Phi(z + \bar{w})$, we anticipate that Φ necessarily has a representation

$$\Phi = \gamma * \left(\frac{\chi_I}{F_T} \right)$$

where $F_T = \gamma * \chi_{(-2T, 2T)}$ and I is some interval. In this case, the natural candidates for limiting point fields again correspond to symmetric intervals $I = (-2T, 2T)$ (or perhaps $I = (-s, s)$ for some suitable value of $s = s(T)$). We are thus led to study the functions of the type

$$H_T(z) := \frac{1}{\sqrt{2\pi}} \int_{-2T}^{2T} \frac{e^{-(z-t)^2/2}}{F_T(t)} dt.$$

The corresponding "1-point function" is then $R_T^h(z) := H_T(z + \bar{z})\chi_{(-T, T)}(\operatorname{Re} z)$.

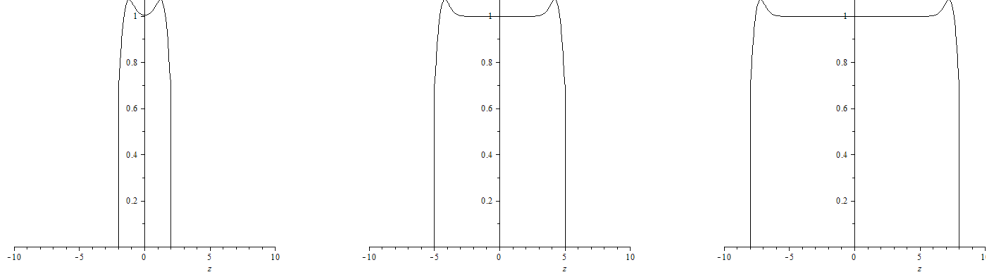


FIGURE 4. The graph of R_T^h restricted to the reals, for $T = 2$, $T = 5$, and $T = 8$.

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YACIN AMEUR, DEPARTMENT OF MATHEMATICS, FACULTY OF SCIENCE, LUND UNIVERSITY, P.O. Box 118, 221 00 LUND, SWEDEN.

E-mail address: `yacin.ameur@maths.lth.se`

NAM-GYU KANG, DEPARTMENT OF MATHEMATICAL SCIENCES, SEOUL NATIONAL UNIVERSITY, SAN56-1 SHINRIM-DONG KWANAK-GU SEOUL 151-747, SOUTH KOREA.

E-mail address: `nkang@snu.ac.kr`

NIKOLAI MAKAROV, DEPARTMENT OF MATHEMATICS, CALIFORNIA INSTITUTE OF TECHNOLOGY, PASADENA, CA 91125, USA.

E-mail address: `makarov@caltech.edu`

ARON WENNMAN, DEPARTMENT OF MATHEMATICS, KTH ROYAL INSTITUTE OF TECHNOLOGY, 100 44 STOCKHOLM, SWEDEN

E-mail address: `aronw@kth.se`